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THE DOUBLE POINTS OF RATIONAL CURVES.

By OSCAR J. PETERSON, University of Wisconsin.

Let P be a point whose homogeneous coördinates are given by the parametric equations:

$$\begin{aligned} \theta x &= f(\lambda) \equiv a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n, \\ (1) \quad \theta y &= g(\lambda) \equiv b_0 \lambda^n + b_1 \lambda^{n-1} + \cdots + b_n, \\ \theta z &= h(\lambda) \equiv c_0 \lambda^n + c_1 \lambda^{n-1} + \cdots + c_n, \end{aligned}$$

where $f(\lambda)$, $g(\lambda)$, $h(\lambda)$ have no common factor. If λ be made to vary continuously from $-\infty$ to $+\infty$, the point P will describe a continuous curve of order n . Such a curve is said to be *rational* or *unicursal*.

If two distinct values s, t of the parameter λ give the same set of coördinates, so that both determine the same point $P_{s, t}$, this point is a *double point*, or singular point of the curve. For these values s, t we have

$$f(s) : g(s) : h(s) = f(t) : g(t) : h(t),$$

or

$$g(s)h(t) - h(s)g(t) = 0,$$

$$h(s)f(t) - f(s)h(t) = 0,$$

$$f(s)g(t) - g(s)f(t) = 0.$$

Each of these equations is divisible by $s - t$, since it is zero for $s = t$. After division by $s - t$ let

$$\begin{aligned} (2) \quad F(s, t) &\equiv \frac{g(s)h(t) - h(s)g(t)}{s - t} = 0, \\ G(s, t) &\equiv \frac{h(s)f(t) - f(s)h(t)}{s - t} = 0, \\ H(s, t) &\equiv \frac{f(s)g(t) - g(s)f(t)}{s - t} = 0 \end{aligned}$$

be the equations. Each pair of values (s, t) which is a solution of the system of equations (2) determines a double point; and, conversely, corresponding to any double point of the curve is a pair of values (s, t) which satisfies equations (2).

(a) If $s \neq t$, the curve crosses itself; $P_{s, t}$ is a *node*, or *crunode*.

(b) If s and t are conjugate imaginaries, and the point $P_{s, t}$ is real, then $P_{s, t}$

is a *conjugate point* (isolated double point, acnode). It is the real intersection of imaginary branches of the curve; there are no other real points in its vicinity.

(c) If two consecutive points of the curve coincide, that is, if there is a solution (s, t) of (2) such that $s = t$, the point is a *cusp* (spinode).

In the following discussion we shall assume that all singular points which occur are either nodes, conjugate points, or cusps.

A theorem of fundamental importance in the study of rational curves is: *Every rational curve of the n th order has $\frac{1}{2}(n-1)(n-2)$ double points.*

This theorem was first proved by Clebsch (1864).¹ The following proof makes no use of Plücker's formulas. It follows immediately from a consideration of equations (2). Thus,

$$\begin{aligned} g(s)h(t) - h(s)g(t) &\equiv (st)^{n-1}[\alpha_{01}(s-t)] \\ &+ (st)^{n-2}[\alpha_{02}(s^2-t^2) + \alpha_{12}(s-t)] \\ &+ (st)^{n-3}[\alpha_{03}(s^3-t^3) + \alpha_{13}(s^2-t^2) + \alpha_{23}(s-t)] \\ &+ \dots + [\alpha_{0n}(s^n-t^n) + \dots + \alpha_{n-1,n}(s-t)], \end{aligned}$$

where $\alpha_{ij} = b_i c_j - c_i b_j$.

$$\begin{aligned} F(s, t) &\equiv (st)^{n-1}[\alpha_{01}] + (st)^{n-2}[\alpha_{02}(s+t) + \alpha_{12}] \\ &+ (st)^{n-3}[\alpha_{03}(s^2+st+t^2) + \alpha_{13}(s+t) + \alpha_{23}] \\ &+ \dots + [\alpha_{0n}(s^{n-1}+s^{n-2}t+\dots+t^{n-1}) + \dots + \alpha_{n-1,n}]. \end{aligned}$$

Since $s^k + s^{k-1}t + \dots + t^k$, where k is a positive integer, can be expressed as a rational integral function of degree k in $s+t$ and st , it is possible to write $F(s, t)$ as a function of degree $n-1$ in $s+t$ and st . Similar expressions are found for $G(s, t)$ and $H(s, t)$.

$$F(s, t) \equiv \bar{F}_{n-1}(s+t, st),$$

$$G(s, t) \equiv \bar{G}_{n-1}(s+t, st),$$

$$H(s, t) \equiv \bar{H}_{n-1}(s+t, st).$$

By Bézout's Theorem there are $(n-1)^2$ solutions $(s+t, st)$ of $\bar{F}_{n-1} = 0$ and $\bar{G}_{n-1} = 0$. But each such pair $s+t, st$ determines only one distinct pair of values s, t ; there are, therefore, precisely $(n-1)^2$ solutions (s, t) of $F = 0$ and $G = 0$. All of these will also satisfy $H = 0$ except those for which both s and t are roots of $h = 0$. The n roots of $h = 0$ can be arranged in pairs in $\frac{1}{2}n(n-1)$ different ways. The number of solutions (s, t) of (2) is

$$(n-1)^2 - \frac{1}{2}n(n-1) = \frac{1}{2}(n-1)(n-2).$$

¹ A. Clebsch, *Crelle's Journ.*, Vol. 64, pp. 43-65. See also J. C. F. Haase, *Math. Ann.*, Vol. 2 (1870), pp. 515-548; and H. Wieleitner, *Theorie der algebraischen Kurven höherer Ordnung*, pp. 74-75. The proof presented here was worked out in connection with a course in higher geometry conducted by Professor Dowling during the year 1916-17.

Hence, if δ be the number of nodes, α of conjugate points, and κ of cusps, we have

$$\delta + \alpha + \kappa = \frac{1}{2}(n-1)(n-2).$$

In general, not all of the $\frac{1}{2}(n-1)(n-2)$ double points can be cusps; *the upper limit of the number of cusps is $\frac{3}{2}(n-2)$* . This may be shown as follows:

In order that a point P of the curve be a cusp it is necessary and sufficient that the value of the parameter λ corresponding to the point P be a common root of the equations $F(\lambda, \lambda) = 0$, $G(\lambda, \lambda) = 0$, $H(\lambda, \lambda) = 0$. [Cf. (2), c.]

$$\begin{aligned} F(\lambda, \lambda) &\equiv h(\lambda)g'(\lambda) - g(\lambda)h'(\lambda),^1 \\ (3) \quad G(\lambda, \lambda) &\equiv f(\lambda)h'(\lambda) - h(\lambda)f'(\lambda), \\ H(\lambda, \lambda) &\equiv g(\lambda)f'(\lambda) - f(\lambda)g'(\lambda). \end{aligned}$$

We notice that

$$\begin{aligned} \rho u &= F(\lambda, \lambda), \\ (4) \quad \rho v &= G(\lambda, \lambda), \\ \rho w &= H(\lambda, \lambda), \end{aligned}$$

are the parametric equations of the curve in homogeneous line coördinates. Each of the polynomials F , G , H is of degree $2(n-1)$ in λ , the coefficient of the highest power obviously vanishing. If there are κ common roots, corresponding to κ cusps,

$$\begin{aligned} F(\lambda, \lambda) &\equiv K(\lambda)\varphi(\lambda), \\ G(\lambda, \lambda) &\equiv K(\lambda)\psi(\lambda), \\ H(\lambda, \lambda) &\equiv K(\lambda)\chi(\lambda), \end{aligned}$$

where K is of degree κ in λ , and φ , ψ , χ are of degree $2(n-1) - \kappa$. The parametric equations of the curve in line coördinates are then

$$\begin{aligned} \rho'u &= \varphi(\lambda), \\ (4') \quad \rho'v &= \psi(\lambda), \\ \rho'w &= \chi(\lambda). \end{aligned}$$

Paralleling the preceding steps, by which we obtained (4) and (4') from (1), we can obtain the parametric equations of the curve in *point coördinates* from (4'). It is not necessary to carry this work through in detail. We must bear

$$\begin{aligned} {}^1 F(\lambda, \lambda) &= \lim_{\epsilon \rightarrow 0} F(\lambda, \lambda + \epsilon) = \lim_{\epsilon \rightarrow 0} \frac{g(\lambda)h(\lambda + \epsilon) - h(\lambda)g(\lambda + \epsilon)}{\lambda - (\lambda + \epsilon)} \\ &= \lim_{\epsilon \rightarrow 0} \frac{g[h + \epsilon h' + \frac{\epsilon^2 h''}{2} + \dots] - h[g + \epsilon g' + \frac{\epsilon^2 g''}{2} + \dots]}{-\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} [h(\lambda)g'(\lambda) - g(\lambda)h'(\lambda) + \epsilon R(\lambda, \epsilon)]. \end{aligned}$$

in mind, however, that the singularities thus brought out are not the point singularities, but the corresponding line singularities: double tangents, isolated double tangents and inflexional tangents. The equations become

$$\sigma x = \Phi(\lambda) \equiv \chi(\lambda)\psi'(\lambda) - \psi(\lambda)\chi'(\lambda),$$

$$\sigma y = \Psi(\lambda) \equiv \varphi(\lambda)\chi'(\lambda) - \chi(\lambda)\varphi'(\lambda),$$

$$\sigma z = X(\lambda) \equiv \psi(\lambda)\varphi'(\lambda) - \varphi(\lambda)\psi'(\lambda).$$

If Φ, Ψ, X have no common factor, the order of the curve is $2[2(n-1) - \kappa - 1]$, this being the degree of Φ, Ψ, X ; if they have ι factors in common, corresponding to ι inflexions, the order is $2[2(n-1) - \kappa - 1] - \iota$. [Cf. the parallel statement, (4) to (4'), above.] The order of the curve is n ; hence

$$n = 2[2(n-1) - \kappa - 1] - \iota,$$

$$(5) \quad 2\kappa + \iota = 3(n-2).$$

(5) indicates that the number of inflexions is odd or even according as n is odd or even; it also yields an upper limit for the number of inflexions,

$$(6) \quad \iota \leq 3(n-2),$$

or of cusps,

$$(7) \quad \kappa \leq \frac{3}{2}(n-2).$$

Wieleitner (*loc. cit.*) obtains this result (7) by means of Plücker's formulas.

When n is odd, the right-hand member of (7) is not an integer; but in this case there is at least one inflexion, and $\frac{3(n-2)-1}{2}$, which is the integral part of the fraction $\frac{3}{2}(n-2)$, gives the greatest possible number of cusps.

For $n > 4$, the maximum number of cusps is less than the total number of double points.

A PROBLEM IN PERSPECTIVE.

By ARNOLD EMCH, University of Illinois.

1. It is well known that any proper plane quadrangle, *i. e.*, a figure determined by four coplanar points of which no three are collinear, may be considered in an infinite number of ways as the perspective of a square, so that to the vertices of the square correspond in a certain order the vertices of the quadrangle. In such a perspective the points within the surface determined in the ordinary sense by the square and the quadrangle do not necessarily have to correspond to each other. For the sake of clearness of representation the figure shows a case of ordinary pictorial perspective in which also the ordinary surfaces correspond to each other. This does not involve loss of generality.